Binomial Theorem and Social Thought

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Abstract

In the 18th century Thomas Bayes with his editor Richard Price applied the binomial theorem to calculating chances. Their predecessor was John Arbuthnot. Arbuthnot concluded that there should have been a divine intervention which kept the constant gender ratio at birth. But there are no needs of consideration except for mere mathematics of chances to find out the reason of the constant ratio if we resort to the central limit theorem of Abraham De Moivre with his friend James Stirling. De Moivre's core insight was to calculate the ratio of the sum of the terms between the two equally distant from the middle term to the sum of all the terms in the given binomial series.

Keywords: gender ratio at birth, binomial theorem, central limit theorem, Bayesian statistics

I. Introduction

It must be well known that John Maynard Keynes (1883-1946) studied mathematics at the University of Cambridge and his long efforts shaped *A Treatise on Probability* (1921) which aimed to bridge a gap between the subjective and the objective by his unique idea of the logical probability. Not only for Keynes but also for the other economic thinkers the problem of chances seems not so minor existence. However, the history in economics or the history of economic thought has not yet fully uncovered the importance of the history of philosophy on probability for economic philosophy. To make a breakthrough this short essay suggests one important path in the history of social thought which runs parallel to the history of economic thought.

II. Smith on Political Arithmetic

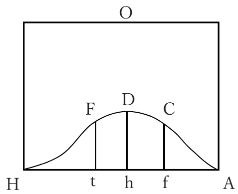
Adam Smith (1723-90) as the founder of political economy or economics said in his *Wealth of Nations* (1776) that "I have no great faith in political arithmetic" (IV.v.b.30) on trade and in his letter to George Chalmers (1742-1825) on 10 November 1785 that "You know that I have little faith in Political Arithmetic" (Corr.249) on population. Smith fiercely criticized Richard Price (1723-91) as "a most superficial Philosopher and by no means an able calculator" in his letter to Chalmers on 22 December of the same year (Corr.251). Price was known as an expert of the reversionary annuities at the time and therefore he was made a kind of scapegoat for Smith's dislike to social statistics.

But Smith's library catalogue compiled probably by himself in 1781 and brought to Tokyo University by Inazo NITOBE (新渡戸稲造) in 1920 shows that Smith had the works by Abraham De

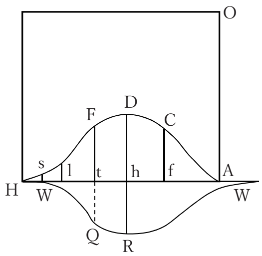
Moivre (1667-1754) and by Thomas Simpson (1710-61) whose mathematics on chance accelerated the development of the social insurance system. The more detailed library catalogue edited by Hiroshi MIZUTA (水田洋) in 2000 shows the interesting point: Smith's library includes both De Moivre (1756) and Simpson (1740) which made the basis of the Bayesian theory. And Price was known as the distinguished editor of the manuscripts written by Thomas Bayes (1701?-61).

Therefore, we could assert that Smith stood very near the innovation or evolution of social sciences in the early modern period but happened to reject the direction to which they began to move. Indeed, Smith sent a letter to Thomas Cadell (1742-1802), bookseller, on 19 June 1784 and ordered volumes 57 and 58 of the *Philosophical Transactions of the Royal Society* and added he had already acquired volumes 56, 59-64 (Corr.239). But those volumes did not contain the essential articles by Bayes and Price which were included in volumes 53-54 published in 1764-65. If Smith had read those articles he would have never resorted to any contemptuous attitudes toward Price.

III. Price as Editor



Here we should understand what Bayes and Price did. A monumental article, Bayes (1764) was edited by Price because when Bayes passed away his relative asked Price to edit his manuscripts. Both men were the ministers of the Presbyterian Church and shared with the thought on chances. Bayes (1764) shows the chance that the probability of an event happening in the next trial should lie between any two degrees of probability (Af/AH and At/AH) as the areal ratio of CftF to ACFH in the diagram above. But this article was criticized for not having the full explanation of the assertions. Therefore Price as the editor took another task for responding the criticism and wrote an article by himself in 1765. Price (1765) shows the way of approximating the areal ratio of CftF to ACFH by equipping the intermediate area RhtQ in the diagram below when the number of trails (n) is very large. The approximation is shown as the inequality



$$\frac{2\Sigma}{1+2Ea^pb^q+\frac{2Ea^pb^q}{n}} < \frac{CftF}{ACFH} < \frac{2\Sigma}{1-2Ea^pb^q-\frac{2Ea^pb^q}{n}}$$

when Σ is RhtQ/ACFH and E is the binomial coefficient for the expansion of $(a+b)^{p+q=n}$. Here $\underline{a}$ means the probability of an event happening (Af/AH) and $\underline{b}$ means the probability not happening (Hf/AH); therefore $a+b=1$. What Price did in the latter article was in short to prepare a robust proof of the Bayesian assertions and to make some technical improvements.

IV. Arbuthnot as Pioneer

Bayes and Price applied the binomial theorem $(a+b)^{p+q}=\sum_{q=0}^{n}\binom{n}{q}a^{p}b^{q}$ (p means the number of times for an event happening and q means the one not happening) to calculating chances. Their predecessor was physician John Arbuthnot (1665-1735) known as one of the best friends of Jonathan Swift (1667-1745), author of *Gulliver's Travels*. Arbuthnot published an article entitled "An Argument for Divine Providence, taken from the constant Regularity observ'd in the Births of both Sexes" on the volume 27 of the *Philosophical Transactions* (1712). He deduced a vivid insight from the data of birth registration in London shown on the table below originally attached to the article.

The Number of Birth Registrations or Christenings in London

Year	Male	Female	M/F	Year	Male	Female	M/F	Year	Male	Female	M/F	Year	Male	Female	M/F
1629	5218	4683	1.114	1650	2890	2722	1.062	1671	6449	6061	1.064	1692	7602	7316	1.039
1630	4858	4457	1.090	1651	3231	2840	1.138	1672	6443	6120	1.053	1693	7676	7483	1.026
1631	4422	4102	1.078	1652	3220	2908	1.107	1673	6073	5822	1.043	1694	6985	6647	1.051
1632	4994	4590	1.088	1653	3196	2959	1.080	1674	6113	5738	1.065	1695	7263	6713	1.082
1633	5158	4839	1.066	1654	3441	3179	1.082	1675	6058	5717	1.060	1696	7632	7229	1.056
1634	5035	4820	1.045	1655	3655	3349	1.091	1676	6552	5847	1.121	1697	8062	7767	1.038
1635	5106	4928	1.036	1656	3668	3382	1.085	1677	6423	6203	1.035	1698	8426	7626	1.105
1636	4917	4605	1.068	1657	3396	3289	1.033	1678	6568	6033	1.089	1699	7911	7452	1.062
1637	4703	4457	1.055	1658	3157	3013	1.048	1679	6247	6041	1.034	1700	7578	7061	1.073
1638	5359	4952	1.082	1659	3209	2781	1.154	1680	6548	6299	1.040	1701	8102	7514	1.078
1639	5366	4784	1.122	1660	3724	3247	1.147	1681	6822	6533	1.044	1702	8031	7656	1.049
1640	5518	5332	1.035	1661	4748	4107	1.156	1682	6909	6744	1.024	1703	7765	7683	1.011
1641	5470	5200	1.052	1662	5216	4803	1.086	1683	7577	7158	1.059	1704	6113	5738	1.065
1642	5460	4910	1.112	1663	5411	4881	1.109	1684	7575	7127	1.063	1705	8366	7779	1.075
1643	4793	4617	1.038	1664	6041	5681	1.063	1685	7484	7246	1.033	1706	7952	7417	1.072
1644	4107	3997	1.028	1665	5114	4858	1.053	1686	7575	7119	1.064	1707	8379	7687	1.090
1645	4047	3919	1.033	1666	4678	4319	1.083	1687	7737	7214	1.072	1708	8239	7623	1.081
1646	3768	3395	1.110	1667	5616	5322	1.055	1688	7487	7101	1.054	1709	7840	7380	1.062
1647	3796	3536	1.074	1668	6073	5560	1.092	1689	7604	7167	1.061	1710	7640	7288	1.048
1648	3363	3181	1.057	1669	6506	5829	1.116	1690	7909	7302	1.083				
1649	3079	2746	1.121	1670	6278	5719	1.098	1691	7662	7392	1.037				

Note: The original style is slightly arranged and added the column M/F but never changed for the numerical data.

For the formula $(M+F)^{n}=\sum_{k=0}^{n}\binom{n}{k}M^{n-k}F^{k}$, if M, F, n each means the probability of male birth (=1), the probability of female birth (=1), and the number of couples who have their babies, the amount the formula contains is equal to 2^{n}. The pattern of sexes among the children can be shown by expanding the formula. When n is equal to 4

$$(M+F)^{4}=\sum_{k=0}^{4}\binom{4}{k}M^{4-k}F^{k}=M^{4}+4M^{3}F+6M^{2}F^{2}+4MF^{3}+F^{4}$$

here M^{4} means all male. And when n is even number the coefficient of the sole middle term is largest because the combination of M^2F^2 has the most patterns.

As $M^{n-k}F^k$ is equal to unity, the ratio of the middle term to the sum of all the terms is equivalent to the ratio for the coefficients and it decreases as n increases. If n is equal to 4 the ratio is 6 over 16 (0.375); if n is equal to 6 it is 5 over 16 (0.3125). Therefore, Arbuthnot asserted as follows: the equality of the numbers of male births and female births is "not Mathematical but Physical" (Arbuthnot 1712, 187). It means that there should have been an invisible intervention. Though the number of births (n) is very large in London the balance between males and females regularly lies in the point on which male birth slightly exceeds female one. This fact is the proof that it is "Art [of Creator], not Chance, that governs" (*Ibid.*, 189).

V. De Moivre as Mathematician

After a few decades De Moivre wrote in the second and third editions of his *Doctrine of Chances* that he could have found the formula which simply expressed the ratio of the middle term to the sum of all the terms of the binomial series by using the way of approximating factorials (n!). His "worthy and learned Friend" James Stirling (1692-1770) had developed the method (De Moivre 1738, 236; 1756, 244). The Stirling's approximation is shown as $n! \sim \left(\frac{n}{e}\right)^n \times \sqrt{2\pi n}$ and when $n = j + k$, the binomial coefficients are $\binom{n}{k} = \frac{n!}{j! \times k!}$. From those assumptions we can deduce the formula

$$\frac{n!}{j! \times k!} = \frac{\left(\frac{n}{e}\right)^n \times \sqrt{2\pi n}}{\left(\frac{j}{e}\right)^j \times \sqrt{2\pi j} \times \left(\frac{k}{e}\right)^k \times \sqrt{2\pi k}} = \frac{n^{n+\frac{1}{2}}}{j^j k^k \sqrt{2\pi jk}}$$

and in case the middle term is examined both j and k are equal to n/2, therefore the binomial coefficient of the middle term is $\frac{n!}{\frac{n}{2}! \times \frac{n}{2}!} = n^{n+\frac{1}{2}} \Big/ \left\{\left(\frac{n}{2}\right)^{n+1} \sqrt{2\pi}\right\} = \frac{2^{n+1}}{\sqrt{2\pi n}}$.

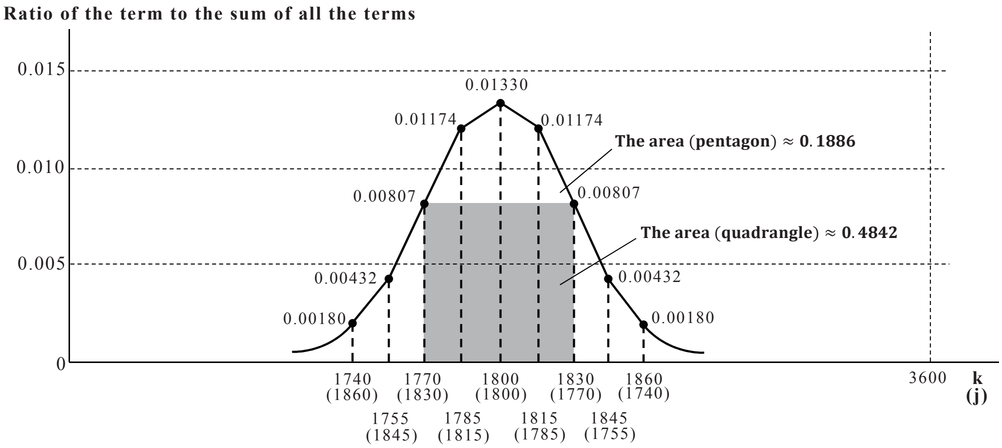
The ratio of the middle term to the sum of all the terms is $\frac{2^{n+1}}{\sqrt{2\pi n}} \times \frac{M^j F^k}{(M+F)^n}$. When M is equal to F, $\frac{M^j F^k}{(M+F)^n} = \frac{F^n}{(2F)^n} = \frac{1}{2^n}$. Accordingly the ratio above is simplified as $\frac{2}{\sqrt{2\pi n}}$, and as n increases, the amount of the ratio decreases. This conclusion is common with Arbuthnot's but there is no need to consider M and F each as equal to 1. In addition, according to De Moivre, the ratio of the sum of the terms in the narrow limits on both sides of the middle term to the sum of all the terms is close to 1 or unity when n is extremely approaching infinity (De Moivre 1738, 243; 1756, 251). De Moivre certainly found out the prototype of the central limit theorem.

De Moivre said that the ratio of "the Sum of the Terms included between two Extreams distant on both sides from the middle Term by an Interval equal to $\frac{1}{2}\sqrt{n}$" to the sum of all the terms was nearly 28 over 41 (De Moivre 1738, 239; 1756, 246-47). If n is equal to 3600, the "Interval" means $\frac{1}{2}\sqrt{3600} = 30$. Therefore, two extremes mean the 1771st as the front term and the 1831st as the last term. Now M is equal to F, j is equal to 1830, and k is equal to 1770, the ratio of the 1771st (1831st) term to the sum of all the terms is:

$$\frac{n^{n+\frac{1}{2}}}{j^j k^k \sqrt{2\pi jk}} \times \frac{M^j F^k}{(M+F)^n} = \binom{3600}{1770} \times \frac{F^{3600}}{(2F)^{3600}} = \frac{3600^{3600+\frac{1}{2}}}{1830^{1830} \times 1770^{1770} \times \sqrt{2\pi \times 1830 \times 1770}} \times \frac{1}{2^{3600}} \approx 0.00807$$

For the middle (1801st) term the ratio is $\frac{2}{\sqrt{2\pi \times 3600}} \approx 0.0133$.

By calculating for the other k (j), we can draw the diagram below. The calculation is rough but the area of both the pentagon and the quadrangle which shows the accumulated ratio is about 0.6728. If the calculation is more exactly (not discretely but continuously) done, the amount becomes very near to 28/41 (0.6829).



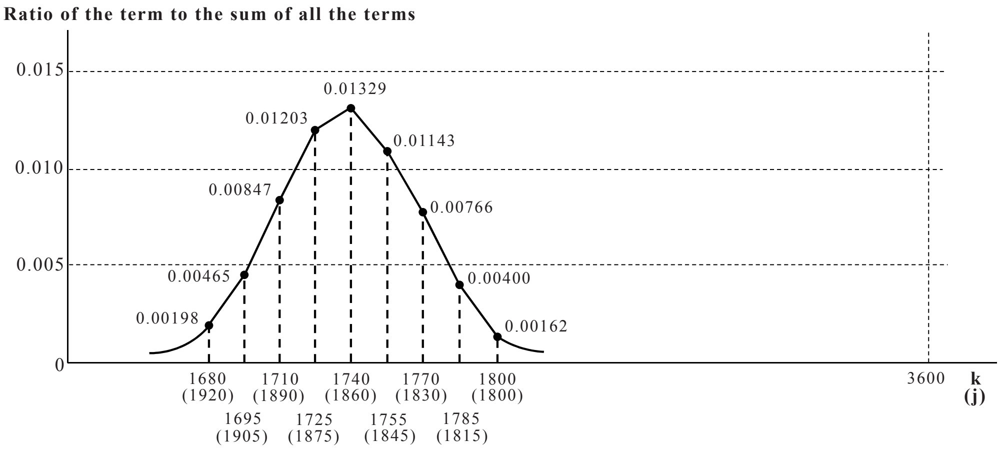
In case n means the number of birth registrations, we can consider k as the number of newborn females. The chance that the number of newborn males or females consists in between 1770 and 1830 is, therefore, almost 70 percent. We can deduce this insight from the ratio of areas. Because the area under the curve in the diagram which means the sum of the ratio each term has to the sum of all the terms is necessarily 1, and the sum of the ratios for any selective terms as certain part of the area under the curve shows the chance on which any particular event with certain range of numerical patterns happens.

The prototype of the central limit theorem suggested by De Moivre was shared with his contemporary mathematician Simpson (Simpson 1740, 76-77). And the idea that the chance of any event could be calculated by the areal ratio was developed by Bayes and Price. Bayes (1764) put the probability of an event along the abscissa, not the ordinate, in order to show the probability distribution and gave the equal chance to each event with certain statistical data; the data means the number of times of happening under the given number of trials. But their core ideas were inherited from De Moivre and Simpson. The Bayesian system is indeed a brainchild of the predecessors.

VI. Conclusion: How to Solve Arbuthnot's Problem

Price wrote in the preface to Bayes (1764) as a letter to natural philosopher John Canton (1718-72) that De Moivre was successful at rebutting those who had "insinuated that the Doctrine of Chances in mathematics is of trivial consequence" by proving that there was certain abductive method to calculate chances. As we have seen their common basic tool was the binomial theorem and the pioneer of applying it to social phenomena was Arbuthnot. Arbuthnot's problem can be solved by De Moivre's central limit theorem when we set the probability M slightly larger than F; because it seems that the middle term with the largest binomial coefficient is not the largest term in reality.

Let us remind of the table attached to Arbuthnot's article. The column M/F shows the ratios of male births to female births over the 82 years; and the average of the ratios is 1.07075. We can assume that M is nearly 7 percent larger than F and can set $\frac{M^jF^k}{(M+F)^n}$ as $\frac{(1.07075F)^jF^k}{(2.07075F)^n} = \frac{1.07075^j}{2.07075^n}$. If n is again equal to 3600, we can draw the diagram below by calculating $\frac{3600^{3600.5}}{j^j \times k^k \times \sqrt{2\pi \times j \times k}} \times \frac{1.07075^j}{2.07075^{3600}}$ for each k (j).



The 1741st term (k=1740) is about the largest; and if we cut out the partial area corresponding to the range between the 1711th (k=1710) and the 1771st term (k=1770) of the binomial series from the total area under the curve which means the accumulated ratio of each term to the sum of all the terms, the former area is roughly calculated to almost 0.6722. Upon this result we can understand as before that the chance that the number of male births consists in between 1830 and 1890 is near to 70 percent. In other words, the chance that the gender ratio (M/F) ranges from 1.034 (1830/1770) to 1.105 (1890/1710) is lying on the same percentage and if we examine how many years in the 82 candidates satisfy the condition we can easily find that in nearly 75 percent (62/82) of those years the range of the gender ratio above is valid.

Therefore, we may conclude that Arbuthnot should have asserted that the constant ratio between the numbers of male births and female births is rather not physical but mathematical because as we have seen M/F can be almost stable within the certain range deduced from the calculation of chances. Keynes also went along with the logical line above and concisely summarized what Arbuthnot should have argued was that “the excess of male births is so invariable, that we may conclude that it is not an even chance whether a male or female be born” (Keynes 1973, 474). But indeed we have to add further comments as follows: on the question of why M is slightly larger than F we cannot explain anything for the details but may only answer to it that there is the experimental process to generate the situation and De Moivre properly described it as the form of the dice predetermined by “some artist” and “not owing to Chance” (De Moivre 1756, 253). Ian Hacking criticizes that even De Moivre equipped with his central limit theorem did appeal to “a Divine hand to work” the “statistical regularity” (Hacking 2006, 171), but in fact De Moivre could carefully distinguish the statistical or mathematical realm from the other domain upon whose horizon the constant law of chances would make itself stand up. Therefore we could still recognize the existence of the preset process and might describe it as “not Mathematical but Physical” with due regard for the pioneers*.

* According to Ritchie and Roser (2019) the natural gender ratio at birth is around 105 males per 100 females and it can range from 103 to 107 per 100. In addition, based on the recent studies they assert that the ratio at conception is equal but during pregnancy the mortality of females exceeds the mortality of males so that the result is male-biased. It depends on a phenomenal explanation but verifies that M/F is naturally expected to be larger than unity and that the average ratio we have deduced from Arbuthnot’s table is almost normal even if it might be affected by male-biased social gender selection. Upon the process we must resort to the physical dimension but upon the result we may calculate chances by a mathematical approach with the probability distribution.

And in this small essay we cannot have examined very extensively the historical contexts in mathematics to which both Arbuthnot (1712) and De Moivre (1738; 1756) belong. But there are a few important studies to be mentioned at least. Todhunter (1865) briefly refers to the objection supposed by Nicolaus Bernoulli (1687-1759) toward Arbuthnot’s emphasis upon the divine intervention. Shoesmith (1987) explains the background of the contemporary debate on Arbuthnot’s influential article and implies that Bernoulli’s approximation which used the summation of the terms of the binomial expansion between the upper and lower numerical limits might prepare the way for the more elaborated solution by De Moivre; and for the better understanding of what Shoesmith suggests we can rely on Hald (2003) as the comprehensive work aiming to explore the mathematical contexts of the matter in the long run.

References

Arbuthnot, J. [1710]1712. An Argument for Divine Providence, Taken from the Constant Regularity Observ'd in the Births of Both Sexes. *Philosophical Transactions*, vol. 27.

Bayes, T. 1764. An Essay towards Solving a Problem in the Doctrine of Chances. *Philosophical Transactions*, vol. 53.

Dale, A. I. 2003. *Most Honourable Remembrance: The Life and Work of Thomas Bayes*. Springer.

Hacking, I. [1975]2006. *The Emergence of Probability: A Philosophical Study of Early Ideas about Probability Induction and Statistical Inference*. Cambridge University Press.

Hald, A. 2003. *A History of Probability and Statistics and Their Applications before 1750*. John Wiley & Sons.

Hayashi, N. 2021-22. Bayes, An Essay towards Solving a Problem in the Doctrine of Chances (1)(2). *OCU Journal of Economics, Management and Information Science*, vols. 21-22. (in Japanese)

Keynes, J. M. [1921]1973. *A Treatise on Probability*. St. Martin's Press and Royal Economic Society.

Mizuta, H. 2000. *Adam Smith's Library: A Catalogue*. Oxford University Press.

Moivre, A. de 1738. *The Doctrine of Chances: Or, a Method of Calculating the Probabilities of Events in Play*, 2nd ed. London.

Moivre, A. de 1756. *The Doctrine of Chances: Or, a Method of Calculating the Probabilities of Events in Play*, 3rd ed. London.

Price, R. 1765. A Demonstration of the Second Rule in the Essay towards the Solution of a Problem in the Doctrine of Chances. *Philosophical Transactions*, vol. 54.

Ritchie, H. and M. Roser. 2019. Gender Ratio. In *Our World in Data*. https://ourworldindata.org/gender-ratio

Shoesmith, E. 1987. The Continental Controversy over Arbuthnot's Argument for Divine Providence. *Historia Mathematica*, vol. 14.

Simpson, T. 1740. *The Nature and Laws of Chance*. London.

Smith, A. [1776]1982. *An Inquiry into the Nature and Causes of the Wealth of Nations*. 2 vols. Liberty Fund.

Smith, A. 1987. *Correspondence of Adam Smith*, revised ed. Liberty Fund.

Todhunter, I. 1865. *A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace*. Macmillan.

Yanaihara, T. 1951. *A Full and Detailed Catalogue of Books which Belonged to Adam Smith*. Iwanami Shoten.