On the Recursion Formula of the Sampling Coefficients on the Compact Semisimple Lie Groups

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Abstract

We present an approach to derive the sampling theorem of the Jacobi transform by using Vretare's method concerning the determination of the Fourier coefficients of the compact semisimple Lie groups. From this, we shall give a recursion formula of the sampling coefficients of the Jacobi transform.

Keywords and phrases: sampling theorem, Jacobi transform, compact symmetric space, the Strum–Liouville boundary valued problem

CONTENTS.

1 Introduction

Sampling theorems are one of the basic tools in communication theory and signal processing. Even now, various types of sampling theorems are obtained in many papers. The Shannon sampling theorem is well known as a fundamental tool. A signal function is called to be band–limited if its band–region is contained in a certain interval. In the terminology of Fourier analysis, the band–limitedness condition is equivalent to the condition that the support of the Fourier transform $\tilde{f}$ of $f \in L^2(\mathbb{R})$ is contained in a certain interval. The Shannon sampling theorem yields that if a function $f \in L^2(\mathbb{R})$ is band–limited, then f can be reconstructed by samples taken at the equidistant sampling points. More preciously, if $f \in L^2(\mathbb{R})$ satisfies supp $\tilde{f} \subseteq [-\pi, \pi]$

then f is reconstructed as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)}. \tag{1.1}$$

This theorem has been generalized in a number of different directions. In one of them, known as Kramer's sampling theorem [7], the kernel of the Fourier transform is replaced by a more general kernel. In paper [11], Zayed studied Kramer's sampling theorems in the case when the kernels arise from the Strum–Liouville boundary valued problems. And from this, he derived various and new types of sampling theorems. In his paper, we are interesting to the sampling theorem deduced by the Jacobi differential equation, which is described below.

Let $\alpha,\ \beta > -1$ and consider the following boundary valued problem for the singular Strum–Liouville differential equation:

$$y'' - \left[\frac{\alpha^2 - 1/4}{4\sin^2(x/2)} + \frac{\beta^2 - 1/4}{4\cos^2(x/2)}\right] y = -\lambda y, \quad (0 < x < \pi)$$

$$|y(0)| < \infty, \qquad |y(\pi)| < \infty.$$

In terms of the Jacobi function, the solution of this problem can be expressed as

$$\phi(x,\lambda) = \left(\sin\frac{x}{2}\right)^{\alpha+1/2}\left(\cos\frac{x}{2}\right)^{\beta+1/2} R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos x), \tag{1.2}$$

where $\gamma = \rho/2$, $\rho = \alpha + \beta + 1$ and

$$R^{(\alpha,\beta)}_t(z) = {}_2F_1\left(-t, t+\rho; \alpha+1; \frac{1-z}{2}\right). \tag{1.3}$$

In [11], Zayed showed the following version of the sampling theorem.

THEOREM 1.1 ([11, Example 4]). *For $f \in L^2(0,\pi)$, its Jacobi transform is defined by*

$$F(\lambda) = \int_0^{\pi} f(x)\left(\sin\frac{x}{2}\right)^{\alpha+1/2}\left(\cos\frac{x}{2}\right)^{\beta+1/2} R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos x)dx. \tag{1.4}$$

Then F is reconstructed by samples as follows:

(1) *When $\gamma \neq 0$, we have*

$$F(\lambda) = \sum_{n=0}^{\infty} F((n+\gamma)^2)\frac{(-1)^{n+1}2(n+\gamma)\Gamma(n+\rho)}{\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})[\lambda-(n+\gamma)^2]\Gamma(n+1)};$$

(2) *When* $\gamma = 0$, *we have*

$$F(\lambda) = F(0)\frac{\sin \pi\sqrt{\lambda}}{\pi\sqrt{\lambda}} + \sum_{n=1}^{\infty} F(n^2)\frac{2\sqrt{\lambda}\sin \pi(\sqrt{\lambda}-n)}{\pi(\lambda - n^2)}.$$

A more general result was obtained by Everitt, Schöttler and Butzer in [4] without assuming the existence of the canonical product of the eigenvalues. And from this, they showed new types of the sampling theorems.

In another direction, the sampling theorems are generalized to the framework of abstract harmonic analysis by replacing $\mathbb{R}$ with a locally compact group. In the case of the locally compact abelian groups, Kluvánek has proved the sampling theorem in [8]. In the non-abelian case, Dooley showed in his paper [1] the sampling theorem for the Cartan motion group by using the techniques of the theory of contraction of Lie group. In [5] Führ and Gröchenig present an approach to derive the sampling theorems on locally compact groups from oscillation estimates.

On the other hand, sampling theorems are studied as relative topics of tomography. In [2] and [3], we study the Fourier reconstruction algorithm and extend this algorithm to the case of Riemannian symmetric spaces. In [2] we fix a K-type δ and give the reconstruction formula for the function of type δ on the Riemannian symmetric space G/K. By using this, the reconstruction formula for the band–limited function can be formally constructed. And in the subsequent paper [3], by taking sampling points suitably, we concretely construct the sampling function of the Radon transform on the complex hyperbolic space. For another example, Stenzel gave the sampling theorem which recovers the rapidly decreasing functions on Riemannian symmetric space from the values of the sampling operator in [9]. He point out his theorem is closest in our papers [2, 3]. We shall discuss in the next paper the relationship between the sampling operator defined by Stenzel and our results in [2, 3].

We shall here describe the context of this paper. In Section 2, for reader's convenience, we give a proof of Theorem 1.1 by using the theory of Everitt, Schöttler and Butzer. In Section 3, applying the theory of Vretare, we construct the recursion formula of the sampling coefficients on the compact semisimple Lie groups.

2 The proof of Theorem 1.1

We here introduce the method of Everitt, Schöttle and Butzer. In [4], they only dealed with the case of the Legendre differential equation, and so we will apply their method to the case of the Jacobi differential equation and gives an elementary proof of Theorem 1.1.

Let $\alpha,\ \beta > -1$ and set $\Delta_{\alpha,\beta}(t) = \sin^{2\alpha+1} t \cos^{2\beta+1} t$. Consider the Jacobi differential equation:

$$-\big(\Delta_{\alpha,\beta}(t)y'\big)' = (4\lambda + \rho^2)\Delta_{\alpha,\beta}(t)y, \ \left(0 < t < \frac{\pi}{2}\right) \tag{2.1}$$

with the boundary valued conditions

$$[y,1](0) = [y,1]\left(\frac{\pi}{2}\right) = 0, \tag{2.2}$$

where $[\cdot,\cdot]$ denotes the bilinear form associated with the differential equation. In this case both endpoints are limit circle and non-oscillatry. In the following, we only consider the case $\gamma \neq 0$, since the case $\gamma = 0$ is reduced to the case of the Legendre differential equation. The pair of the fundamental solutions of (2.1) is given by

$$\left\{ R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos 2t), R^{(\beta,\alpha)}_{\sqrt{\lambda}-\gamma}(-\cos 2t) \right\}.$$

Here $R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}$ denote the Jacobi function described in (1.3). For the sake of simplicity, we put

$$\begin{aligned} R(t) &= R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos 2t) = {}_2F_1(\gamma + \sqrt{\lambda}, \gamma - \sqrt{\lambda}; \alpha + 1; \sin^2 t), \\ S(t) &= R^{(\beta,\alpha)}_{\sqrt{\lambda}-\gamma}(-\cos 2t) = {}_2F_1(\gamma + \sqrt{\lambda}, \gamma - \sqrt{\lambda}; \beta + 1; \cos^2 t). \end{aligned}$$

And we set

$$\varphi_1(t) = \frac{(R(t) - S(t))\Gamma(\gamma + \sqrt{\lambda})\Gamma(\gamma - \sqrt{\lambda})}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)}, \tag{2.3}$$

$$\varphi_2(t) = \frac{(R(t) + S(t))\Gamma(\gamma + \sqrt{\lambda})\Gamma(\gamma - \sqrt{\lambda})}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)}. \tag{2.4}$$

After these preparations, the Kramer type kernel $K(x,\lambda)$ on $[0,\pi/2]$ is generated by

$$K(x,\lambda) = [\varphi_1, 1](0)\varphi_2(x) - [\varphi_2, 1](0)\varphi_1(x). \tag{2.5}$$

A direct computation implies

$$
\begin{aligned}
[\varphi_1, 1](0) &= -\lim_{t\to 0} \Delta_{\alpha,\beta}(t)\varphi_1'(t) \\
&= -\lim_{t\to 0} \frac{(\gamma^2-\lambda)\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})\sin^{2\alpha+2} t\cos^{2\beta+2} t}{2(\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad \times \left\{ {}_2F_1(\gamma+1+\sqrt{\lambda}, \gamma+1-\sqrt{\lambda}; \alpha+2; \sin^2 t) \right. \\
&\quad \left. + {}_2F_1(\gamma+1+\sqrt{\lambda}, \gamma+1-\sqrt{\lambda}; \beta+2; \cos^2 t) \right\} \\
&= \lim_{t\to 0} \frac{(\gamma^2-\lambda)\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})\cos^{2\beta+2} t}{2(\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad \times {}_2F_1(\gamma+1+\sqrt{\lambda}, \gamma+1-\sqrt{\lambda}; \beta+2; \cos^2 t) \\
&= \frac{(\gamma^2-\lambda)\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})}{2(\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad \times {}_2F_1(\beta-\gamma+1-\sqrt{\lambda}, \beta-\gamma+1+\sqrt{\lambda}; \beta+2; 1) \\
&= \frac{(\gamma^2-\lambda)\Gamma(\beta+2)\Gamma(\alpha+1)\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})}{2(\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1+\sqrt{\lambda})\Gamma(\gamma+1-\sqrt{\lambda})} \\
&= \frac{1}{2}.
\end{aligned}
$$

By the same computation as above, we also have

$$[\varphi_2, 1](0) = -\frac{1}{2}.$$

Substituting these into (2.5), we have

$$K(x, \lambda) = R(t) = R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos 2t).$$

Therefore we can get the integral transform

$$F(\lambda) = \int_0^{\pi/2} f(t) R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos 2t)\Delta_{\alpha,\beta}(t)dt. \tag{2.6}$$

Similarly, the interpolation function $G(\lambda)$ is given by

$$\begin{aligned}
G(\lambda) &= [K,1]\left(\frac{\pi}{2}\right) \\
&= -\frac{2(\gamma^2-\lambda)}{\alpha+1}\lim_{t\to\pi/2}\sin^{2\alpha+2}t\cos^{2\beta+2}t \\
&\qquad\qquad \times {}_2F_1(\gamma+1+\sqrt{\lambda},\gamma+1-\sqrt{\lambda};\alpha+2;\sin^2 t) \\
&= -\frac{2(\gamma^2-\lambda)}{\alpha+1}\lim_{t\to\pi/2}\sin^{2\alpha+2}t \\
&\qquad\qquad \times {}_2F_1(\alpha-\gamma+1-\sqrt{\lambda},\alpha-\gamma+1+\sqrt{\lambda};\alpha+2;\sin^2 t) \\
&= \frac{-2\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})}. \qquad (2.7)
\end{aligned}$$

From the expression of (2.7), the zeroes of $G(\lambda)$ are $\lambda_n = (\gamma+n)^2$ for $n \in \mathbb{Z}_{\geqslant 0}$ and Theorem 1.1 in [4] yields that λ_n are taken at the samples of the integral transform (2.6).

We next compute the sampling function of (2.6). We have from (2.7) that

$$G'(\lambda) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\psi(\gamma+\sqrt{\lambda})-\psi(\gamma-\sqrt{\lambda}))}{\sqrt{\lambda}\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})},$$

where ψ denotes the polygamma function. We obtain upon taking the limit $\lambda \to \lambda_n = (\gamma+n)^2$ that

$$\begin{aligned}
G'(\lambda_n) &= \lim_{\lambda\to\lambda_n} G'(\lambda) \\
&= \frac{(-1)^n\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+1)}{(n+\gamma)\Gamma(n+\rho)}. \qquad (2.8)
\end{aligned}$$

Consequently, the sampling function of (2.6) can be written by

$$\frac{G(\lambda)}{G'(\lambda_n)(\lambda-\lambda_n)} = \frac{(-1)^{n+1}2(n+\gamma)\Gamma(n+\rho)}{\Gamma(n+1)\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})(\lambda-(n+\gamma)^2)},$$

from which we can get the assertion of Theorem 1.1.

3 Notation for Lie groups and root systems

Let $\mathfrak{g}$ be a semisimple Lie algebra on $\mathbb{R}$ and let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. The Killing form $\langle\cdot,\cdot\rangle$ induces an inner product on $\mathfrak{a}$. Let $\mathfrak{h} \supset \mathfrak{a}$ be a Cartan subalgebra and put $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$. We fix an ordering in $\mathfrak{a}^*$ and denote by Σ^+ the set of positive restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{g}$. We set $\rho = \frac{1}{2}\sum_{\mu\in\Sigma^+} m(\mu)\mu$, where $m(\mu)$ denotes the multiplicity of μ.

We write $\mathfrak{u}$ for the corresponding compact real form of $\mathfrak{g}$, that is, $\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Denote by G_c the simply connected Lie group with Lie algebra $\mathfrak{g}$ and by G, K and U the analytic subgroups of G_c with their Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{u}$, respectively. This permit us to identify the irreducible finite dimensional representations of G_c with those of G and U. We remark that U is a maximal compact subgroup of G_c.

Let $A = \exp\mathfrak{a}$. We put $\mathfrak{t} = \mathfrak{h}_{\mathfrak{k}} + \sqrt{-1}\mathfrak{a}$. Fix an ordering in $\sqrt{-1}\mathfrak{t}$ which is compatible with the one on $\mathfrak{a}$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}$. We normalized the Haar measure on U so that the total measure on U is 1 and denote it by du.

Let $G = KAN$ be an Iwasawa decomposition of G. For $g \in G$, we decompose as $g = \kappa(g)\exp H(g)n(g)$, where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. For $\lambda \in \mathfrak{a}^*$, let $\mathscr{H}^\lambda = L^2(K/M)$ and define the action of G on $\mathscr{H}^\lambda$ by

$$(\pi_\lambda(g)\varphi)(k) = e^{(\sqrt{-1}\lambda-\rho)H(g^{-1}k)}\varphi(\kappa(g^{-1}k)).$$

$(\pi_\lambda, \mathscr{H}^\lambda)$ is called the spherical principal series representation on G. The zonal spherical function φ_λ is given by

$$\varphi_\lambda(g) = \int_K e^{(\sqrt{-1}\lambda-\rho)(H(g^{-1}k))}dk.$$

Let $(\pi_\Lambda, \mathscr{H}^\Lambda)$ denote the finite dimensional irreducible representation on U with highest weight $\Lambda \in \mathfrak{t}^*$. As shown in [6, Theorem 4.1, p.535], Λ is characterized as the following conditions:

$$\Lambda|_{\mathfrak{h}_{\mathfrak{k}}} = 0, \qquad \frac{\langle\Lambda,\mu\rangle}{\langle\mu,\mu\rangle} \in \mathbb{Z}_{\geqslant 0} \qquad (\mu \in \Sigma^+). \tag{3.1}$$

We write for Δ the set of $\Lambda \in \mathfrak{t}^*$ satisfying the relations (3.1). By means of (3.1), we look upon Λ as an element in $\mathfrak{a}_c^*$. It is also known (see for instance [10]) that

$$\int_U \Phi_{\Lambda_1}(x)\overline{\Phi_{\Lambda_2}(x)}dx = \begin{cases} 0 & (\Lambda_1 \neq \Lambda_2) \\ \frac{1}{d_{\Lambda_1}} & (\Lambda_1 = \Lambda_2) \end{cases}. \tag{3.2}$$

Here d_Λ is expressed as

$$d_\Lambda = \frac{c(\sqrt{-1}\rho)c(-\sqrt{-1}\rho)}{c(\sqrt{-1}(\rho+\Lambda))c(-\sqrt{-1}(\rho+\Lambda))}, \tag{3.3}$$

$c(\lambda)$ denoting the Harish-Chandra c-function.

As mentioned above, the irreducible finite dimensional irreducible representations on U can be regarded as the representations on G and G_c and we simply write them for the same symbol π_Λ. We put

$$\Omega = \left\{ H \in \mathfrak{a}; |\mu(H)| < \frac{\pi}{2} \text{ for any } \mu \in \Sigma^+ \right\}.$$

Then as pointed out in [10, p. 353], we have that

$$\Phi_\Lambda(\exp \sqrt{-1}H) = \varphi_{-\sqrt{-1}(\Lambda+\rho)}(\exp H) \tag{3.4}$$

for $H \in \Omega$. Since the Iwasawa projection extends holomorphically from $G \exp(\sqrt{-1}\Omega)K_c$ to $\mathfrak{a}_c$, the zonal spherical function φ_λ can be regarded as a K-biinvarinat smooth function on U. And we denote it by φ_λ again.

Let $f \in L^2(K\backslash U/K)$ and define the integration transform of f with respect to the kernel φ_λ as

$$F(\lambda) = \int_U f(u)\varphi_\lambda(u)du. \tag{3.5}$$

Remark. From (3.4), if $\lambda = -\sqrt{-1}(\Lambda + \rho)$ then the integration transform (3.5) coincides with the Fourier transform on the compact symmetric space $K\backslash U/K$.

Expanding φ_λ in the Fourier series on $K\backslash U/K$

$$\begin{aligned} \varphi_\lambda(u) &= \sum_{\Lambda\in\Delta} d_\Lambda c_\Lambda \Phi_\Lambda(u), \\ c_\Lambda &= \int_U \varphi_\lambda(u)\overline{\Phi_\Lambda(u)}du \end{aligned} \tag{3.6}$$

and substituting (3.6) into (3.5), we have

$$\begin{aligned} F(\lambda) &= \int_U f(u)\varphi_\lambda(u)du \\ &= \sum_{\Lambda\in\Delta} d_\Lambda c_\Lambda \int_U f(u)\Phi_\Lambda(u)du \\ &= \sum_{\Lambda\in\Delta} d_\Lambda c_\Lambda \int_U f(u)\varphi_{-\sqrt{-1}(\Lambda+\rho)}(u)du \\ &= \sum_{\Lambda\in\Delta} d_\Lambda c_\Lambda F(-\sqrt{-1}(\Lambda+\rho)). \end{aligned} \tag{3.7}$$

In this way, we can get a sampling expansion of F. We call c_Λ the sampling coefficients on the compact symmetric space $K\backslash U/K$.

In the remainder of this section, we suppose that $\operatorname{rank}\mathfrak{g} = 1$. Let μ denote the unique simple root of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We identify $\mathfrak{a}_c^*$ with $\mathbb{C}$ via the correspondence $\lambda \mapsto \langle\lambda,\mu\rangle/\langle\mu,\mu\rangle$. For $\lambda \in \mathfrak{a}_c^*$, we define $H_\lambda \in \mathfrak{a}_c$ by $\mu(H_\lambda) = \langle\mu,\lambda\rangle$ and from this, we identify $\mathfrak{a}_c^*$ with $\mathfrak{a}_c$ via the correspondence $\lambda \mapsto H_\lambda$.

We define the values of α and β by the table below:

G	U	K	α	β	ρ
$\mathbb{R}$	$SO(2)$	$\{1\}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
$Spin(n,1)$ $(n \geqslant 2)$	$Spin(n+1)$	$Spin(n)$	$\frac{n}{2}-1$	$-\frac{1}{2}$	$\frac{n-1}{2}$
$SU(1,1)$	$SU(2)$	$U(1)$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$SU(n,1)$ $(n \geqslant 2)$	$SU(n+1)$	$S(U(n)\times U(1))$	$n-1$	0	n
$Sp(n,1)$	$Sp(n+1)$	$Sp(n)\times Sp(1)$	$2n-1$	1	$2n+1$
$F_{4(-20)}$	$F_{4(-52)}$	$Spin(9)$	7	3	11

Under these parametrization, the following proposition holds.

Proposition 3.1 ([6, Theorem 4.5, p. 543]). Φ_Λ *is expressed as follows:*

(1) *When* $\beta = -1/2$, $\Lambda = n\mu$ *and*

$$\Phi_{n\mu}(\exp H) = P_n^{(\alpha,\alpha+1)}(\cos\mu(H));$$

(2) *When* $\beta \neq -1/2$, $\Lambda = 2n\mu$ *and*

$$\Phi_{2n\mu}(\exp H) = P_n^{(\alpha,\beta)}(\cos 2\mu(H)).$$

Here

$$P_n^{(\alpha,\beta)}(z) = {}_2F_1\left(-n, n+\rho; \alpha+1; \frac{1-z}{2}\right)$$

is a Jacobi polynomial of degree (α,β).

As is well known, the Harish-Chandra c-function is expressed as

$$c^{(\alpha,\beta)}(\lambda) = \frac{2^{\rho-\sqrt{-1}\lambda}\Gamma(\alpha+1)\Gamma(\sqrt{-1}\lambda)}{\Gamma\left(\frac{1}{2}(\sqrt{-1}\lambda+\rho)\right)\Gamma\left(\frac{1}{2}(\sqrt{-1}\lambda+\alpha-\beta+1)\right)}. \tag{3.8}$$

We use (3.8) to compute (3.3) and hence we obtain the following:

When $\beta = -1/2$,

$$d_{n\mu} = \frac{(2\alpha+2n+1)n!\Gamma(2\alpha+n+1)}{\Gamma(2\alpha+2)}; \tag{3.9}$$

When $\beta \neq -1/2$,

$$d_{2n\mu} = \frac{(\rho+2n)\Gamma(\rho+n)\Gamma(\beta+1)\Gamma(\alpha+n+1)}{n!\Gamma(\rho+1)\Gamma(\beta+n+1)\Gamma(\alpha+1)}. \tag{3.10}$$

We also note that

$$\int_U du = \int_0^{\pi/2} \Delta_{\alpha,\beta}(t)dt = \frac{1}{2}B(\alpha+1,\beta+1).$$

We also know from [6, p. 543] that the zonal spherical function φ_λ is expressed as

$$\varphi_\lambda(\exp H) = {}_2F_1\left(\frac{1}{2}(\rho-\sqrt{-1}\lambda), \frac{1}{2}(\rho+\sqrt{-1}\lambda); \alpha+1; -\sinh^2\mu(H)\right).$$

We thus have for $H \in \Omega$ that

$$\varphi_\lambda(\exp\sqrt{-1}H) = {}_2F_1\left(\frac{1}{2}(\rho-\sqrt{-1}\lambda), \frac{1}{2}(\rho+\sqrt{-1}\lambda); \alpha+1; \sin^2\mu(H)\right).$$

And from this, we see that

$$\varphi_\lambda(\exp tH) = R^{(\alpha,\beta)}_{\sqrt{-1}\lambda/2-\gamma}(\cosh 2t), \quad \left(\gamma = \frac{\rho}{2}\right). \tag{3.11}$$

Remark. Comparing (3.11) with (2.6), we need to make a change of variable λ to $-2\sqrt{-1}\sqrt{\lambda}$ for the proof of Theorem 1.1.

4 The recursion formula of the sampling coefficients

We keep the notation in the previous section and the assumption that $\operatorname{rank}\mathfrak{g} = 1$. When restricted attention to the rank one case, the sampling coefficients can be directly computed by using Green's integral formula. However for the generalization to the higher rank case, we give a group theoretic interpretation of Theorem 1.1. In order to compute the sampling coefficients, we need the following proposition due to Vretare.

PROPOSITION 4.1 ([10, Theorem 4.8]). *We set $\mu_0 = 2\mu$ when $\beta \neq -\frac{1}{2}$ and $\mu_0 = \mu$ when $\beta = -\frac{1}{2}$. Then we have for $\lambda \in \mathfrak{a}^*$ that*

$$\varphi_{-\sqrt{-1}(\mu_0+\rho)}(g)\varphi_\lambda(g) = d_{\mu_0}(\lambda)\varphi_{\lambda-\sqrt{-1}\mu_0}(g) + d_{\mu_0}(-\lambda)\varphi_{\lambda+\sqrt{-1}\mu_0}(g) + d_0(\lambda)\varphi_\lambda(g),$$

where

$$d_{\mu_0}(\lambda) = \frac{c(-\sqrt{-1}(\mu_0 + \rho))c(-\sqrt{-1}\lambda)}{c(-\sqrt{-1}(\mu_0 + \lambda))} \tag{4.1}$$

and

$$d_0(\lambda) = 1 - d_{\mu_0}(\lambda) - d_{\mu_0}(-\lambda). \tag{4.2}$$

In the remainder of this section we only consider the case $\beta \neq -1/2$. For the case $\beta = -1/2$, we only need to change $2n$ to n in the following computations. Applying Proposition 4.1 repeatedly, we see that there exist the functions $d_{n,k}(\lambda)$, $(k = -n, -n+1, \ldots, n-1, n)$ such that

$$\varphi_{-\sqrt{-1}(2n\mu+\rho)}(g)\varphi_{-2\sqrt{-1}\sqrt{\lambda}}(g) = \sum_{k=-n}^{n} d_{n,k}(\lambda)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g). \tag{4.3}$$

Remark. From the explicit expression of the Harish-Chandra c-function (3.8), it is easy to see that the coefficients $d_{n,k}(\lambda)$ are rational functions on λ.

For brevity we set $d_{n,k}(\lambda) = 0$ when $|k| > n$. With the help of the expression (3.4), (3.9) and (4.3), we can compute the sampling coefficients $c_{2n\mu}$ defined in (3.6). Indeed, we have

$$\begin{aligned} c_{2n\mu} &= \int_U \varphi_{-2\sqrt{-1}\sqrt{\lambda}}(u)\overline{\Phi_{2n\mu}(u)}du \\ &= \int_U \varphi_{-2\sqrt{-1}\sqrt{\lambda}}(u)\varphi_{-\sqrt{-1}(2n\mu+\rho)}(u)du \qquad (4.4)\\ &= \sum_{k=-n}^{n} \frac{2d_{n,k}(\lambda)}{B(\alpha+1,\beta+1)} \int_0^{\pi/2} R^{(\alpha,\beta)}_{\sqrt{\lambda}+k\mu-\gamma}(\cos 2t)\Delta_{\alpha,\beta}(t)dt. \qquad (4.5)\end{aligned}$$

By using the formula

$$\int_0^{\pi/2} R^{(\alpha,\beta)}_{\sqrt{\lambda}+k\mu-\gamma}(\cos 2t)\Delta_{\alpha,\beta}(t)dt = \frac{2\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\gamma+1+k+\sqrt{\lambda})\Gamma(\gamma+1-k+\sqrt{\lambda})},$$

we have from (4.5) that

$$c_{2n\mu} = \sum_{k=-n}^{n} \frac{4\Gamma(\rho+1)d_{n,k}(\lambda)}{\Gamma(\gamma+1+k+\sqrt{\lambda})\Gamma(\gamma+1-k+\sqrt{\lambda})}. \tag{4.6}$$

Consequently, for getting the explicit expressions of the sampling coefficients, it is sufficient to compute the values of $d_{n,k}(\lambda)$. We perform this to deduce the recursion formula for $d_{n,k}(\lambda)$.

We here directly compute (4.4) and give the explicit expression of $c_{2n\mu}$. From Proposition 3.1 and (3.11), (4.4) is written as

$$c_{2n\mu} = \frac{2}{B(\alpha+1,\beta+1)} \int_0^{\pi/2} R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos 2t) R^{(\alpha,\beta)}_{n\mu}(\cos 2t)\Delta_{\alpha,\beta}(t)dt. \qquad (4.7)$$

We set

$$R(t) = R^{(\alpha,\beta)}_{\sqrt{\lambda}-\gamma}(\cos 2t), \quad P(t) = R^{(\alpha,\beta)}_{n\mu}(\cos 2t).$$

We have from Green's integral formula that

$$(\lambda - (n+\gamma)^2) \int_0^{\pi/2} R(t)P(t)\Delta_{\alpha,\beta}(t)dt = [R,P]\left(\frac{\pi}{2}\right). \qquad (4.8)$$

By a straightforward calculation, we obtain

$$\begin{aligned} &\Delta_{\alpha,\beta}(t)R(t)P'(t) \\ &\quad = \frac{-n(n+\rho)}{\alpha+1}\Delta_{\alpha,\beta}(t)\,{}_2F_1(\gamma+\sqrt{\lambda},\gamma-\sqrt{\lambda};\alpha+1;\sin^2 t) \\ &\qquad \times {}_2F_1(-n+1,n+\rho+1;\alpha+2;\sin^2 t) \\ &\quad = \frac{-n(n+\rho)}{\alpha+1}\sin^{2\alpha+1} t\,{}_2F_1(\gamma+\sqrt{\lambda},\gamma-\sqrt{\lambda};\alpha+1;\sin^2 t) \\ &\qquad \times {}_2F_1(n+\alpha+1,-n-\beta;\alpha+2;\sin^2 t) \end{aligned}$$

and thus

$$\lim_{t\to\pi/2} \Delta_{\alpha,\beta}(t)R(t)P'(t) = 0.$$

Similarly we obtain

$$\begin{aligned} &\Delta_{\alpha,\beta}(t)R'(t)P(t) \\ &\quad = \frac{2(\gamma^2-\lambda)}{\alpha+1}\Delta_{\alpha,\beta}(t)\,{}_2F_1(\gamma+1+\sqrt{\lambda},\gamma+1-\sqrt{\lambda};\alpha+2;\sin^2 t) \\ &\qquad \times {}_2F_1(-n,n+\rho;\alpha+1;\sin^2 t) \\ &\quad = \frac{2(\gamma^2-\lambda)}{\alpha+1}\sin^{2\alpha+1} t\,{}_2F_1(\alpha-\gamma+1-\sqrt{\lambda},\alpha-\gamma+1+\sqrt{\lambda};\alpha+2;\sin^2 t) \\ &\qquad \times {}_2F_1(-n,n+\rho;\alpha+1;\sin^2 t) \end{aligned}$$

and thus

$$\lim_{t\to\pi/2} \Delta_{\alpha,\beta}(t)R'(t)P(t) = \frac{(-1)^n 2\Gamma(\alpha+1)^2\Gamma(n+\beta+1)}{\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})\Gamma(\alpha+n+1)}.$$

Collecting these, we see that (4.8) yields

$$(\lambda - (n+\gamma)^2) \int_0^{\pi/2} R(t)P(t)\Delta_{\alpha,\beta}(t)dt = \frac{(-1)^{n+1}2\Gamma(\alpha+1)^2\Gamma(\beta+n+1)}{\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})\Gamma(\alpha+n+1)}.$$

Therefore (4.7) implies

$$c_{2n\mu} = \frac{(-1)^{n+1}4\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\beta+n+1)}{\Gamma(\gamma+\sqrt{\lambda})\Gamma(\gamma-\sqrt{\lambda})\Gamma(\alpha+n+1)\Gamma(\rho+1)(\lambda-(n+\gamma)^2)}.$$

Let us return our attention to the computation of the recursion formula of $d_{n,k}(\lambda)$. We have from Proposition 4.1 that

$$\begin{aligned}\varphi_{-\sqrt{-1}(2\mu+\rho)}(g)\varphi_{-2\sqrt{-1}\sqrt{\lambda}}(g) &= d_{2\mu}(-2\sqrt{-1}\sqrt{\lambda})\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+\mu)}(g) \\ &+ d_{2\mu}(2\sqrt{-1}\sqrt{\lambda})\varphi_{-2\sqrt{-1}(\sqrt{\lambda}-\mu)}(g) + d_0(-2\sqrt{-1}\sqrt{\lambda})\varphi_{-2\sqrt{-1}\sqrt{\lambda}}(g).\end{aligned}$$

We use (3.8) to compute (4.1) and (4.2) and immediately obtain that

$$d_{2\mu}(2\sqrt{-1}\sqrt{\lambda}) = \frac{(\gamma+\sqrt{\lambda})(\gamma_-+\sqrt{\lambda})(\rho+1)}{2\sqrt{\lambda}(2\sqrt{\lambda}+1)(\alpha+1)}, \tag{4.9}$$

$$d_0(2\sqrt{-1}\sqrt{\lambda}) = \frac{2(\alpha-\beta)(\gamma^2-\lambda)}{(1-4\lambda)(\alpha+1)}, \tag{4.10}$$

where $\gamma_- = (\alpha-\beta+1)/2$. For simplicity, we set

$$\begin{aligned} d_+(k) &= d_{2\mu}(-2\sqrt{-1}(\sqrt{\lambda}+k\mu)), \\ d_-(k) &= d_{2\mu}(2\sqrt{-1}(\sqrt{\lambda}+k\mu)), \\ d_0(k) &= d_0(-2\sqrt{-1}(\sqrt{\lambda}+k\mu)). \end{aligned}$$

From these, we see that

$$d_{1,-1}(\lambda) = d_+(1), \quad d_{1,1}(\lambda) = d_-(1), \quad d_{1,0}(\lambda) = d_0(1). \tag{4.11}$$

In particular, taking $\lambda = (n+\gamma)^2$, we have from (4.9) and (4.10) that

$$\begin{aligned} d_2((n+\gamma)^2) &= \frac{(n+\rho)(\alpha+n+1)(\rho+1)}{(2n+\rho)(2n+\rho+1)(\alpha+1)}, \\ d_2(-(n+\gamma)^2) &= \frac{n(\beta+n)(\rho+1)}{(2n+\rho)(2n+\rho-1)(\alpha+1)}, \\ d_0((n+\gamma)^2) &= \frac{2(\alpha-\beta)n(n+\rho)}{((2n+\rho)^2-1)(\alpha+1)}. \end{aligned}$$

Hereafter we simply write $d_+ = d_2(-(n+\gamma)^2)$, $d_- = d_2((n+\gamma)^2)$, $d_0 = d_0((n+\gamma)^2)$.

THEOREM 4.2. *Retain the above notation. the following recursion formula holds for $d_{n,k}(\lambda)$.*

(1) *When $n = 1$,*

$$d_{1,-1}(\lambda) = d_+(1),\ d_{1,1}(\lambda) = d_-(1),\ d_{1,0}(\lambda) = d_0(1),$$

where

$$d_+(1) = \frac{(\gamma + \sqrt{\lambda})(\gamma_- + \sqrt{\lambda})(\rho + 1)}{2\sqrt{\lambda}(2\sqrt{\lambda} + 1)(\alpha + 1)},$$
$$d_0(1) = \frac{2(\alpha - \beta)(\gamma^2 - \lambda)}{(1 - 4\lambda)(\alpha + 1)};$$

(2) *When $n > 1$, $d_{n,k}(\lambda)$ are determined recursively as follows:*

$$d_+ d_{n+1,k}(\lambda) + d_- d_{n-1,k}(\lambda)$$
$$= d_+(k+1) d_{n,k-1}(\lambda) + d_-(k-1) d_{n,k+1}(\lambda) + d_0(k) d_{n,k}(\lambda) - d_0 d_{n,k}(\lambda)$$

Proof. The case when $n = 1$ is already shown in (4.9) and (4.10). We suppose that $n > 1$ and compute $\varphi_{-\sqrt{-1}(2\mu+\rho)}(g)\varphi_{-\sqrt{-1}(2n\mu+\rho)}(g)\varphi_{-2\sqrt{-1}\sqrt{\lambda}}(g)$ by the different two ways. From Proposition 4.1, we have

$$\varphi_{-\sqrt{-1}(2\mu+\rho)}(g)\varphi_{-\sqrt{-1}(2n\mu+\rho)}(g) = d_+\varphi_{-\sqrt{-1}(2n\mu+2\mu+\rho)}(g)$$
$$+ d_-\varphi_{-\sqrt{-1}(2n\mu-2\mu+\rho)}(g) + d_0\varphi_{-\sqrt{-1}(2n+\rho)}(g).$$

And thus we have

$$\varphi_{-\sqrt{-1}(2\mu+\rho)}(g)\varphi_{-\sqrt{-1}(2n\mu+\rho)}(g)\varphi_{-2\sqrt{-1}\sqrt{\lambda}}(g)$$
$$= \sum_{k=-n-1}^{n+1} d_+ d_{n+1,k}(\lambda)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g)$$
$$+ \sum_{k=-n+1}^{n-1} d_- d_{n-1,k}(\lambda)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g)$$
$$+ \sum_{k=-n}^{n} d_0 d_{n,k}(\lambda)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g).$$

Secondly using the equation in (4.3), we have

$$\begin{aligned}
&\varphi_{-\sqrt{-1}(2\mu+\rho)}(g)\varphi_{-\sqrt{-1}(2n\mu+\rho)}(g)\varphi_{-2\sqrt{-1}\sqrt{\lambda}}(g)\\
&\qquad = \varphi_{-\sqrt{-1}(2\mu+\rho)}(g)\left\{\sum_{k=-n}^{n} d_{n,k}(\lambda)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g)\right\}\\
&\qquad = \sum_{k=-n}^{n} d_{n,k}(\lambda)\Big\{d_+(k)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+(k+1)\mu)}(g)\\
&\qquad + d_-(k)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+(k-1)\mu)}(g) + d_0(k)\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g)\Big\}\\
&\qquad = \sum_{k=-n-1}^{n+1} \Big\{d_+(k-1)d_{n,k-1}(\lambda) + d_-(k+1)d_{n,k+1}(\lambda)\\
&\qquad + d_0(k)d_{n,k}(\lambda)\Big\}\varphi_{-2\sqrt{-1}(\sqrt{\lambda}+k\mu)}(g).
\end{aligned}$$

Compared term by term, we can immediately obtain the desired results. □

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